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Exact solution and asymptotic behaviour of the asymmetric simple exclusion process on a ring

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Abstract

In this paper, we study an exact solution of the asymmetric simple exclusion process on a periodic lattice of finite sites with two typical updates, i.e., random and parallel. Then, we find that the explicit formulae for the partition function and the average velocity are expressed by the Gauss hypergeometric function. In order to obtain these results, we effectively exploit the recursion formula for the partition function for the zero-range process. The zero-range process corresponds to the asymmetric simple exclusion process if one chooses the relevant hop rates of particles, and the recursion gives the partition function, in principle, for any finite system size. Moreover, we reveal the asymptotic behaviour of the average velocity in the thermodynamic limit, expanding the formula as a series in system size.

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1. Introduction

The asymmetric simple exclusion process (ASEP) is an exemplar of stochastic many-particle systems and provides fundamental models for various collective phenomena within the framework of nonequilibrium statistical mechanics [1–4]. With some particular conditions, the ASEP shows a wide variety of nonequilibrium critical phenomena such as boundary-induced phase transitions [5], spontaneous symmetry breaking [6] and phase separation [7–9]. As described here, the ASEP has been extensively studied both with periodic boundaries and with open boundaries, and at the same time one may consider several updates such as random and parallel [10]. In this paper, we focus on the ASEP on a lattice of finite sites with the periodic boundary condition and then consider a discrete-time evolution both with random dynamics and with parallel dynamics. The ASEP with the periodic boundary condition is suitable for applying a theoretical approach for the first time.

In the ASEP, particles exclusively occupy sites on a lattice and hop from site to site in a definite direction with a constant probability, i.e., they move as if being driven by a biased field. The particles have a short-range interaction due to the exclusive occupation (i.e., the hard-core exclusion). However, as far as the parallel dynamics is concerned, it may induce a long-range interaction as the density of particles grows. (Note that the parallel update treats all the sites equally, and the ASEP with parallel update is hence regarded as a stochastic cellular automaton.)

Generally in stochastic models, an asymmetric dynamics causes asymmetric transition rates between states of configuration and it is manifested as a current of probability [11]. According to that current, one observes a macroscopic phenomenon such as a density flow, and the system is hence thought to be far from equilibrium without any equilibrium state. However, nonequilibrium systems may have a steady state with a constant flow of particles, i.e., a nonequilibrium steady state [12]. The ASEP and the zero-range process, both of which are the subjects of the present paper, are typical examples for that kind of nonequilibrium system. We shall investigate the nonequilibrium steady state of these stochastic dynamical systems.

Analytic methods were improved especially in the context of transport phenomena, e.g., traffic flow, and then exact solutions were presented by powerful theoretical methods such as the matrix-product ansatz [13] and the cluster-approximation method [14]. However, the previous results are for systems in the thermodynamic limit, i.e., of infinite system size. As far as nonequilibrium many-particle systems are concerned, it is essential that one should investigate these systems of finite size and then consider them in the thermodynamic limit. In this paper, we present an explicit formula of the partition function for the ASEP on a ring of finite sites, which allows for an exact calculation of physical values.

We find that the partition function is expressed by the Gauss hypergeometric function. In order to obtain this result, we exploit the recursion formalism for the partition function for the zero-range process (ZRP) [15], which is given in [16]. As described later, the ASEP corresponds to the ZRP if one chooses the relevant hop rates of particles. Since the partition function for the ZRP is, in principle, calculated for any finite system size via the recursion, one can obtain that for the ASEP in the same way.

As an application of the present result, we calculate an explicit formula for the average velocity of particles in the ASEP. Due to the formulae for the hypergeometric function, we can provide an explicit formula for the average velocity by the Gauss hypergeometric function as well as for the partition function. In addition to that, we provide the series expansion of the average velocity in system size by solving a sequence of equations in order from the lowest. The series expansion reveals the asymptotic behaviour of a physical value in the thermodynamic limit. In particular, we recover the well-known result given in [14].

In most recent works [17, 18], the same formulae as we obtain in the present work are given by using the Bethe ansatz solution of the master equation for the ZRP. They find that the hop rates required by the applicability of the Bethe ansatz allow two parameters and in a special case the ZRP reduces to the ASEP. It is interesting that, although the ZRP is concerned in common, the two ways seem entirely different until just before reaching the same formula. We remark that, in contrast with the Bethe ansatz, our approach can be generalized to provide an exact solution of the ZRP if one takes other hop rates which do not comply with the conditions required by the Bethe ansatz [19].

This paper is organized as follows. In section 2, we define the ZRP and see that in a special case the ZRP corresponds to the ASEP. Then, we derive the partition function for the

ASEP after preparing the general formalism for that for the ZRP. In section 3, we provide the series expansion of the average velocity in system size. Section 4 is devoted to conclusions and final remarks. Long derivations of formulae are included in the appendix.

2. Exact solution of the asymmetric simple exclusion process

2.1. The zero-range process

The zero-range process is an exactly solvable stochastic process and it has been widely used as a model for systems of many particles interacting through a short-range interaction [15]. Particles in the ZRP are indistinguishable, occupying sites on a lattice (any lattice in any dimension), and each lattice site may contain an integer number of particles. These particles hop to the next sites with a rate which depends on the number of particles at the departure site. One is to apply the ZRP to phenomenological studies choosing a relevant hop rate of particles which determines the microscopic behaviour.

One of the most distinct properties of the ZRP is that its steady state is given by a factorized form [15]. Each of these factors of the steady state are for each site of the lattice and are determined by the hop rates. In this sense, these factors are called the single-site weights. We remark that the steady-state behaviour is determined by the single-site weights and consequently one may infer the hop rates after choosing the single-site weights as desired.

The ZRP in one dimension can be mapped onto the ASEP if one rearranges particles and sites in the ZRP on a lattice in the ASEP in order that site m, containing n_m particles in the ZRP, should be thought of as, in the ASEP, the mth particle occupying a site and unoccupied n_m sites, i.e., the distance to the $(m-1)$th particle in front of the mth particle. In particular, the ZRP corresponds to the ASEP if one chooses the relevant hop rates (defined by (14)). In recent works [20, 21], we proposed a traffic-flow model connecting the ASEP and the ZRP with an additional parameter.

In what follows, we consider the ZRP in one dimension with N particles on a periodic lattice containing M sites labelled $l = 1, 2, \ldots, M$. It corresponds to the ASEP with M particles on the periodic lattice of L $(=M + N)$ sites, and then one should understand that the system is of size L and the density is $\rho = M/L$.

2.2. The partition function for the ZRP

In this subsection, following [15], we define the partition function for the ZRP and then find that the generating function for the partition function is obtained from that for the single-site weights.

The nonequilibrium steady-state probability $P(\{n_m\})$ of finding the system in a configuration $\{n_m\} = \{n_1, n_2, \ldots, n_M\}$ is given as a product of the single-site weights denoted by $f(n)$:

$$P(\{n_m\}) = \frac{1}{Z_{M,N}} \prod_{m=1}^{M} f(n_m) \qquad (n_1 + n_2 + \cdots + n_M = N), \tag{1}$$

where $Z_{M,N}$, a normalization, is the so-called partition function. The probability that a given site (e.g., site 1) contains n particles is given by

$$p(n) = \sum_{n_2+n_3+\cdots+n_M=N-n} P(\{n, n_2, \ldots, n_M\}) = f(n)\frac{Z_{M-1,N-n}}{Z_{M,N}}. \tag{2}$$

The sum of $p(n)$ over n is unity by definition. Thus, we obtain the recursion formula for the partition functions,

$$Z_{M,N} = \sum_{n=0}^{N} f(n) Z_{M-1,N-n}, \tag{3}$$

$$Z_{1,k} = f(k) \qquad (k \geqslant 1). \tag{4}$$

Note that this is a recursion for a double series with respect to M and N, and the partition functions are calculated recursively from the initial values $Z_{1,k}$.

Considering the generating functions $\widehat{f}(\zeta) := \sum_{n=0}^{\infty} f(n)\zeta^n$ and $\widehat{Z}_M(\zeta) := \sum_{n=0}^{\infty} Z_{M,n}\zeta^n$, we have a recursion with respect only to M: $\widehat{Z}_M(\zeta) = \widehat{f}(\zeta)\widehat{Z}_{M-1}(\zeta)$. Accordingly, we find the fundamental relation

$$\widehat{Z}_M(\zeta) = (\widehat{f}(\zeta))^M. \tag{5}$$

Thus, as a matter of form the partition function $Z_{M,N}$ for any N is obtained from the single-site weights $f(n)$.

The single-site weights, being obtained from the hop rates of particles, entirely characterize the zero-range process. In [15, 16], two formulae of the single-site weights, respectively, corresponding to random and parallel updates are given as follows. For random update, the single-site weights $f(n)$ are expressed as

$$f(0) = 1, \qquad f(n) = \prod_{j=1}^{n} \frac{1}{u(j)} \quad (n \geqslant 1), \tag{6}$$

where $u(n)$ are the hop rates of particles when the departure site contains n particles. Note that $u(0) = 0$ by definition. For parallel update, the single-site weights are expressed as

$$f(n) = \begin{cases} 1 - u(1) & (n = 0) \\ \dfrac{1-u(1)}{1-u(n)} \displaystyle\prod_{j=1}^{n} \dfrac{1-u(j)}{u(j)} & (n \geqslant 1). \end{cases} \tag{7}$$

It is remarkable that according to the update the single-site weights have different recursions. From (6) one finds that the single-site weights for random update have a recursion

$$u(n) f(n) = f(n-1). \tag{8}$$

In a similar fashion, from (7) we have the recursion that the single-site weights for parallel update satisfy

$$u(n+1) f(n+1) = f(n) - u(n) f(n). \tag{9}$$

We remark that as for the single-site weights the recursions (8) and (9) may identify the different updates instead of the explicit formulae (6) and (7), which will be seen in the course of calculations on the average velocity of particles in the ZRP.

2.3. The average velocity of particles in the ZRP

In this subsection, we shall provide the explicit formulae of the average velocity (or the mean hop rate averaged in the nonequilibrium steady state) of particles in the ZRP with random and parallel updates. In general, the flux of a transport system briefly presents its macroscopic property, and especially in nonequilibrium systems it is one of the few quantitative criteria. In the present study, the flux of particles is obtained by multiplying the average velocity by the density of particles, since the number of particles conserves due to the periodic boundary condition.

The average velocity, denoted by $v_{M,N}$, is defined by

$$v_{M,N} = \sum_{n=0}^{N} u(n)p(n) = \sum_{n=0}^{N} u(n)f(n)\frac{Z_{M-1,N-n}}{Z_{M,N}}. \tag{10}$$

As mentioned in subsection 2.2, the recursion for the single-site weights play the central role in the subsequent calculations.

We first consider the ZRP with random update. From (3), (8) and (10), we find that the average velocity is expressed by the partition function, i.e.,

$$v_{M,N} = \frac{Z_{M,N-1}}{Z_{M,N}}. \tag{11}$$

Next, we consider the parallel update. From (3), (9) and (10), we find that the average velocity satisfies the recursion with respect to N:

$$v_{M,N+1}Z_{M,N+1} = Z_{M,N} - v_{M,N}Z_{M,N}. \tag{12}$$

(It is suggestive that (12) is corresponding to (9).) Solving (12) for $v_{M,N}$ with respect to N, the average velocity is expressed by the partition function, i.e.,

$$v_{M,N} = -\frac{\sum_{n=0}^{N-1}(-1)^n Z_{M,n}}{(-1)^N Z_{M,N}}. \tag{13}$$

2.4. *Exact solution of the ASEP*

Now, we turn to the calculations for the partition function and the average velocity in the ASEP. As mentioned in subsection 2.1, one chooses the hop rates in the ZRP as

$$u(0) = 0, \qquad u(n) = p \quad (0 < p < 1, n \geqslant 1), \tag{14}$$

and the ZRP is thereby conformed to the ASEP with hop rate p.

2.4.1. Random update. First, we consider the ASEP with random update. From (6), the single-site weights are $f(n) = p^{-n} (n \geqslant 0)$ and then the generating function for them becomes

$$\widehat{f}(\zeta) = \sum_{n=0}^{\infty}\left(\frac{\zeta}{p}\right)^n = \frac{1}{1-\frac{\zeta}{p}}. \tag{15}$$

Hence, from (5) the generating function for the partition functions is figured out as

$$\widehat{Z}_M(\zeta) = \left(1-\frac{\zeta}{p}\right)^{-M} = \sum_{n=0}^{\infty}\frac{(M)_n}{p^n n!}\zeta^n. \tag{16}$$

Thus, we obtain the partition function for the ASEP with random update,

$$Z_{M,N} = \frac{(M)_N}{p^N N!} = \frac{1}{p^N}\binom{M+N-1}{M-1}, \tag{17}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. Therefore, from (11) and (17) the average velocity for the ASEP with random update is given by

$$v_{M,N} = \frac{Np}{M+N-1}. \tag{18}$$

2.4.2. Parallel update. Next, we consider the ASEP with parallel update. From (5) and (7), the generating function for the partition function becomes

$$\widehat{Z}_M(\zeta) = \left(1 - p + \frac{(1-p)\zeta}{p - (1-p)\zeta}\right)^M. \tag{19}$$

Expanding (19) in a power series of ζ, we obtain the partition function for the ASEP with parallel update,

$$Z_{M,0} = (1-p)^M, \tag{20}$$

$$Z_{M,N} = \frac{(-1)^N(-p)^M M}{1-p} F\left({M+1, N+1 \atop 2}; \frac{1}{1-p}\right) \qquad (N \geqslant 1), \tag{21}$$

where

$$F\left({\alpha, \beta \atop \gamma}; z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \tag{22}$$

is the Gauss hypergeometric function. (See the appendix for details of the above calculations.) This presentation allows one to take advantage of the formulae for the hypergeometric functions to make advanced calculations.

In order to calculate the average velocity according to (13), we carry out the sum in the numerator beforehand. Using the Gauss recursion formula with respect to parameters

$$\frac{\alpha z}{\gamma} F\left({\alpha+1, \beta+1 \atop \gamma+1}; z\right) = F\left({\alpha, \beta+1 \atop \gamma}; z\right) - F\left({\alpha, \beta \atop \gamma}; z\right), \tag{23}$$

one finds

$$\sum_{n=0}^{N-1} (-1)^n Z_{M,n} = (-p)^M F\left({M, N \atop 1}; \frac{1}{1-p}\right). \tag{24}$$

Finally, the average velocity for the ASEP with parallel update is obtained as

$$v_{M,N} = \frac{p-1}{M} \frac{F\left({M, N \atop 1}; \frac{1}{1-p}\right)}{F\left({M+1, N+1 \atop 2}; \frac{1}{1-p}\right)}. \tag{25}$$

3. Asymptotic behaviour of the exact solutions

In this section, we investigate the asymptotic behaviour of the exact solutions of the ASEP, i.e., we find the power series expansion of the partition function $Z_{M,N}$ and the average velocity $v_{M,N}$, for a given density $\rho = M/L$, with respect to the system size $L\,(=M+N)$. As mentioned in the introduction, the average velocity is a physical value and is hence expanded in a power series with respect to the system size. In contrast, the partition function shall be expanded as an asymptotic series.

3.1. Random update

First, we consider the random update. After being expressed with the system size L, the average velocity (11) is easily expanded in a power series as follows:

$$v_{M,N} = \frac{p(1-\rho)L}{L-1} = p(1-\rho)(1 + L^{-1} + L^{-2} + \cdots). \tag{26}$$

From (26), we have an expected consequence

$$\lim_{L\to\infty} v_{M,N} = p(1-\rho). \tag{27}$$

3.2. Parallel update

Next, we consider the parallel update. For the sake of convenience, we shall present the power expansion of the average velocity before turning to the partition function. To begin with, we change the independent variable as $z = 1/(1-p)$. Then, using the formula for the derivative of the hypergeometric function:

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[z^{\gamma-1}(1-z)^{\alpha+\beta-\gamma}F\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};z\right)\right] \tag{28}$$

$$= (\gamma-n)_n z^{\gamma-n-1}(1-z)^{\alpha+\beta-\gamma-n}F\left(\begin{matrix}\alpha-n,\beta-n\\ \gamma-n\end{matrix};z\right), \tag{29}$$

we can transform the rhs of (25) into a logarithmic differentiation:

$$v_{M,N} = \frac{z-1}{M}\frac{\mathrm{d}}{\mathrm{d}z}\log\left(z(1-z)^{M+N}F\left(\begin{matrix}M+1,N+1\\ 2\end{matrix};z\right)\right). \tag{30}$$

The Gauss hypergeometric differential equation for the hypergeometric function in the argument of the logarithm in (30) is given by

$$z(1-z)\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + [2-(M+N+3)z]\frac{\mathrm{d}w}{\mathrm{d}z} - (M+1)(N+1)w = 0. \tag{31}$$

Accordingly, the argument of the logarithm also satisfies a differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + \frac{1-M-N}{z-1}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{MN}{z(z-1)}w = 0, \tag{32}$$

and we thus find that the average velocity $v_{M,N}$ is the solution of a Riccati equation

$$\frac{p(p-1)}{M}\frac{\mathrm{d}}{\mathrm{d}p}v_{M,N} = v_{M,N}^2 - \left(1+\frac{N}{M}\right)v_{M,N} + \frac{Np}{M}. \tag{33}$$

Expanding the average velocity as $v_{M,N} = v_0(\rho) + v_1(\rho)L^{-1} + v_2(\rho)L^{-2} + \cdots$, we separate the Riccati equation according to power of L:

$$v_0^2 - \frac{1}{\rho}v_0 + \frac{p(1-\rho)}{\rho} = 0, \tag{34}$$

$$p(p-1)\frac{\mathrm{d}}{\mathrm{d}p}v_{j-1} = \sum_{\substack{k+l=j\\ k,l\geqslant 0}}\rho v_k v_l - v_j \qquad (j\geqslant 1). \tag{35}$$

These sequential equations, solved in turn starting from v_0, give the power series expansion of the average velocity $v_{M,N}$ with respect to the system size L:

$$v_{M,N} = \frac{1-\sqrt{1-4p\rho(1-\rho)}}{2\rho} + \frac{(1-\rho)p(1-p)}{1-4p\rho(1-\rho)}L^{-1} + \frac{(1-\rho)p(1-p)[1-2p+p(3p+1)\rho(1-\rho)]}{[1-4p\rho(1-\rho)]^{5/2}}L^{-2} + \cdots. \tag{36}$$

In particular, we recover the well-known result for the average velocity in the thermodynamic limit [14], i.e.,

$$\lim_{L\to\infty} v_{M,N} = \frac{1-\sqrt{1-4p\rho(1-\rho)}}{2\rho}. \tag{37}$$

4. Conclusion and remark

In the present paper, we provide an exact solution of the ASEP on a periodic lattice with respect to two typical dynamics, i.e., random update and parallel update. To begin with, we focus on the following two facts; first, the ZRP corresponds to the ASEP if one takes the hop rates of particles to be a constant in the ZRP. Second, the partition function of the ZRP is exactly solvable in the sense that the partition function for any system size is obtained by a recursive calculation. Then, by solving the recursion with the relevant hop rates, we obtain the partition function for the ASEP.

For the random dynamics, it is given by the product of a power of the hop rate and the binomial coefficient in which the number of sites and that of particles appear. For the parallel dynamics, it is given by the Gauss hypergeometric function in which the number of sites and that of particles equally appear as the parameters. It is remarkable that the hop rate becomes the independent variable. Thus, the mathematical structure of the ASEP becomes clear from the viewpoint of special functions.

By using the above results, we calculate the average velocities and obtain their series expansions in system size. The series expansions confirm the previous results given in the thermodynamic limit. Note that, if necessary, one can obtain the higher order correction terms for the average velocity in the parallel dynamics by solving the sequential equations in order.

From the present results, it is expected that one can estimate the partition function for a ZRP with other hop rates than those we take in this paper, which will be reported in the near future.

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Appendix A. Calculation on the generating function for the ASEP

From the hop rates (14), the single-site weights are

$$f(0) = 1 - p, \qquad f(n) = \left(\frac{1-p}{p}\right)^n \qquad (n \geqslant 1). \tag{A.1}$$

For simplicity, we let $\gamma = 1 - p$ and $\beta = (1-p)/p$ hereafter. Then, the generating function for the single-site weights is

$$\widehat{f}(\zeta) = \gamma + \sum_{n=1}^{\infty} (\beta\zeta)^n = \gamma + \frac{\beta\zeta}{1-\beta\zeta}. \tag{A.2}$$

Accordingly, the generating function for the partition function becomes

$$\widehat{Z}_M(\zeta) = \left(\gamma + \frac{\beta\zeta}{1-\beta\zeta}\right)^M = \sum_{k=0}^{M} \binom{M}{k} \gamma^{M-k} \left(\frac{\beta\zeta}{1-\beta\zeta}\right)^k. \tag{A.3}$$

Using the Euler transformation for a series

$$\sum_{k=0}^{\infty} a_k z^k = \frac{1}{1+z} \sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n b_n \qquad \text{where} \quad b_n := \sum_{r=0}^{n} \binom{n}{r} a_r, \tag{A.4}$$

we obtain

$$\widehat{Z}_M(\zeta) = (1-\beta\zeta)\sum_{n=0}^{\infty} c_n(\beta\zeta)^n = c_0 + \sum_{n=1}^{\infty}(c_n - c_{n-1})(\beta\zeta)^n, \tag{A.5}$$

where

$$c_n = \sum_{r=0}^{n}\binom{n}{r}\binom{M}{r}\gamma^{M-r}. \tag{A.6}$$

From (A.6), we have

$$c_n - c_{n-1} = \sum_{r=1}^{n}\binom{n-1}{r-1}\binom{M}{r}\gamma^{M-r} \tag{A.7}$$

$$= M\sum_{r=0}^{N-1}\frac{(1-N)_r(1-M)_r}{(2)_r(1)_r}\gamma^{M-r-1} \tag{A.8}$$

$$= \gamma^{M-1}MF\left(\begin{matrix}1-M,1-N\\2\end{matrix};\gamma^{-1}\right). \tag{A.9}$$

After all, we obtain (19) from (A.5) and (A.9).

Moreover, using the Kummer's transformation formula

$$F\left(\begin{matrix}\alpha,\beta\\\gamma\end{matrix};z\right) = (1-z)^{\gamma-\alpha-\beta}F\left(\begin{matrix}\gamma-\alpha,\gamma-\beta\\\gamma\end{matrix};z\right), \tag{A.10}$$

one obtains (19) from (21).

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